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NEW COMBINATORIAL INTERPRETATIONS OF RAMANUJAN'S PARTITION CONGRUENCES MOD 5, 7 AND 11

F. G. GARVAN

ABSTRACT. Let $p(n)$ denote the number of unrestricted partitions of n. The congruences referred to in the title are $p(5n+4)$, $p(7n+5)$ and $p(11n+6) \equiv 0$ (mod 5, 7 and 11, respectively). Dyson conjectured and Atkin and Swinnerton-Dyer proved combinatorial results which imply the congruences mod 5 and 7. These are in terms of the rank of partitions. Dyson also conjectured the existence of a "crank" which would likewise imply the congruence mod 11. In this paper we give a crank which not only gives a combinatorial interpretation of the congruence mod 11 but also gives new combinatorial interpretations of the congruences mod 5 and 7. However, our crank is *not* quite what Dyson asked for; it is in terms of certain restricted triples of partitions, rather than in terms of ordinary partitions alone.

Our results and those of Dyson, Atkin and Swinnerton-Dyer are closely related to two unproved identities that appear in Ramanujan's "lost" notebook. We prove the first identity and show how the second is equivalent to the main theorem in Atkin and Swinnerton-Dyer's paper. We note that all of Dyson's conjectures mod 5 are encapsulated in this second identity. We give a number of relations for the crank of vector partitions mod 5 and 7, as well as some new inequalities for the rank of ordinary partitions mod 5 and 7. Our methods are elementary relying for the most part on classical identities of Euler and Jacobi.

1. Introduction. Let $p(n)$ denote the number of unrestricted partitions of n. Ramanujan discovered and later proved

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

For elementary proofs of (1.1) and (1.2) see Ramanujan [20]. The most elementary proof of (1.3) is due to Winquist [24]. Much more is known than (1.1)–(1.3). In fact, for $\alpha \geqslant 1$

$$p(5^\alpha n + \delta_{5,\alpha}) \equiv 0 \pmod{5^\alpha}, \tag{1.4}$$

$$p(7^\alpha n + \delta_{7,\alpha}) \equiv 0 \pmod{7^{[(\alpha+2)/2]}}, \tag{1.5}$$

$$p(11^\alpha n + \delta_{11,\alpha}) \equiv 0 \pmod{11^\alpha}. \tag{1.6}$$

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Here $\delta_{t,\alpha}$ is the reciprocal modulo t^α of 24. (1.4) and (1.5) were first proved by G. N. Watson [**23**] in 1938. For an elementary proof of (1.4) see Hirschhorn and Hunt [**16**] and for an elementary proof of (1.5) see Garvan [**14**]. (1.6) was proved by A. O. L. Atkin [**11**] in 1967.

In 1944 F. J. Dyson [**13**] discovered empirically some remarkable combinatorial interpretations of (1.1) and (1.2). Dyson defined the *rank* of a partition as the largest part minus the number of parts. For example, the partition 4 + 4 + 3 + 2 + 1 + 1 + 1 has rank 4 − 7 = −3. Let $N(m, n)$ denote the number of partitions of n with rank m and let $N(m, t, n)$ denote the number of partitions of n with rank congruent to m modulo t. Dyson conjectured that

$$(1.7) \quad N(0,5,5n+4) = N(1,5,5n+4) = \cdots = N(4,5,5n+4) = \frac{p(5n+4)}{5}$$

and

$$(1.8) \quad N(0,7,7n+5) = N(1,7,7n+5) = \cdots = N(6,7,7n+5) = \frac{p(7n+5)}{7}.$$

(1.7) and (1.8) were later proved by A. O. L. Atkin and H. P. F. Swinnerton-Dyer [**7**] in 1953. These are the combinatorial interpretations of (1.1) and (1.2). Atkin and Swinnerton-Dyer's proof is analytic, relying heavily on the properties of modular functions. No combinatorial proof is known. All that is known combinatorially about the rank is that

$$(1.9) \qquad N(m,n) = N(-m,n),$$

which follows from the fact that the operation of conjugation reverses the sign of the rank. A trivial consequence is that

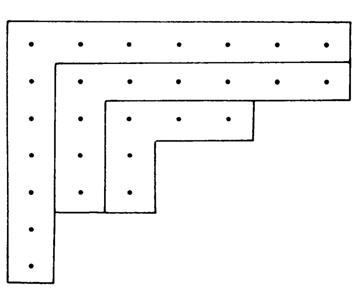
$$(1.10) \qquad N(m,t,n) = N(t-m,t,n).$$

Atkin and Swinnerton-Dyer's paper contains proofs of other conjectures of Dyson for the rank such as

$$(1.11) \qquad N(1,5,5n+1) = N(2,5,5n+1).$$

As well they calculate the generating functions for $N(a,t,tn+k) - N(b,t,tn+k)$ for $t = 5, 7$ and all possible values of a, b and k. Later, Atkin and Hussain [**8**] do the same for $t = 11$ and in 1965 O'Brien [**19**] does the same for $t = 13$.

It is worth noting that Atkin [**9**] has generalized Dyson's rank. Any partition may be represented as a set of nested right angles of nodes. The partition 7 + 7 + 5 + 3 + 3 + 1 + 1 is represented by three such right angles:



If π is a partition, Atkin defines $d_i(\pi)$ as the number of nodes in the horizontal part of the ith right angle in the graph of π minus the number of nodes in the vertical part of the right angle, so that $d_1(\pi)$ is Dyson's rank and $d_i(\pi) = 0$ if π does not have an ith right angle. The $d_i(\pi)$ are called the *successive ranks* of π. Atkin gives alternate combinatorial interpretations of (1.1) and (1.2) that are analogous to (1.7) and (1.8). Namely, if we denote by $N^*(m,n)$ (resp. $N^*(m,t,n)$) the number of partitions π of n in which $d_1(\pi) - 2d_2(\pi) = m$ (resp. the number of partitions π of n in which $d_1(\pi) - 2d_2(\pi) \equiv m \pmod t$) then

(1.12)
$$N^*(0,5,5n+4) = N^*(1,5,5n+4) = \cdots = N^*(4,5,5n+4) = \frac{p(5n+4)}{5}$$

and

(1.13)
$$N^*(0,7,7n+5) = N^*(1,7,7n+5) = \cdots = N^*(6,7,7n+5) = \frac{p(7n+5)}{7}.$$

Atkin defines an operation C_i of i-conjugacy which acts on partitions and satisfies

(1.14) $$d_1(C_i\pi) = d_1(\pi) - 2d_i(\pi).$$

Hence,

(1.15) $$N^*(m,n) = N(m,n)$$

and (1.12) and (1.13) follow trivially from (1.7) and (1.8). Atkin's successive ranks have been studied further by Andrews [**2**, §9.3].

The result analogous to (1.7) and (1.8) for the prime 11 does not hold. Dyson conjectured the existence of what he called the "crank" that satisfies

(1.16) $$M(m,t,n) = M(t-m,t,n),$$

(1.17) $$M(0,11,11n+6) = M(1,11,11n+6) = \cdots$$
$$= M(10,11,11n+6) = p(11n+6)/11$$

and as well as other relations for the crank modulo 11, where $M(m,t,n)$ denotes the number of partitions of n with crank congruent to m modulo t. Many have searched in vain for Dyson's crank. We provide a combinatorial interpretation of (1.3). No combinatorial interpretation of (1.3) has hitherto been found. We also provide new interpretations of (1.1) and (1.2) (see (1.27) and (1.28) below). We have *not* discovered Dyson's elusive crank. Our main result (see (1.29) below) does not actually divide up the partitions of $11n+6$ into 11 equal classes but rather it gives a combinatorial interpretation of $p(11n+6)/11$ in terms of the crank of what we call vector partitions.

To describe our main result we need some more notation. For a partition, π, let $\#(\pi)$ be the number of parts of π and $\sigma(\pi)$ be the sum of the parts of π (or the number π is partitioning) with the convention $\#(\phi) = \sigma(\phi) = 0$ for the empty partition, ϕ, of 0. Let

$$V = \{(\pi_1, \pi_2, \pi_3) \mid \pi_1 \text{ is a partition into distinct parts and } \pi_2, \pi_3 \text{ are unrestricted partitions}\}.$$

We shall call the elements of V *vector partitions*. For $\pi = (\pi_1, \pi_2, \pi_3)$ in V we define the sum of parts, s, a weight, ω, and a crank, r, by

$$s(\pi) = \sigma(\pi_1) + \sigma(\pi_2) + \sigma(\pi_3), \tag{1.18}$$

$$\omega(\pi) = (-1)^{\#(\pi_1)}, \tag{1.19}$$

$$r(\pi) = \#(\pi_2) - \#(\pi_3). \tag{1.20}$$

We say π is a vector partition of n if $s(\pi) = n$. For example, if $\pi = (5 + 3 + 2, 2 + 2 + 1, 2 + 1 + 1)$ then $s(\pi) = 19$, $\omega(\pi) = -1$, $r(\pi) = 0$ and π is a vector partition of 19. The number of vector partitions of n (counted according to the weight ω) with crank m is denoted by $N_V(m, n)$, so that

$$N_V(m, n) = \sum_{\substack{\pi \in V \\ s(\pi) = n \\ r(\pi) = m}} \omega(\pi). \tag{1.21}$$

The number of vector partitions of n (counted according to the weight ω) with crank congruent to k modulo t is denoted by $N_V(k, t, n)$, so that

$$N_V(k, t, n) = \sum_{m=-\infty}^{\infty} N_V(mt + k, n) = \sum_{\substack{\pi \in V \\ s(\pi) = n \\ r(\pi) \equiv k \pmod t}} \omega(\pi). \tag{1.22}$$

By considering the transformation that interchanges π_2 and π_3 we have

$$N_V(m, n) = N_V(-m, n) \tag{1.23}$$

so that

$$N_V(t - m, t, n) = N_V(m, t, n). \tag{1.24}$$

We have the following generating function for $N_V(m, n)$:

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m, n) z^m q^n = \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - zq^n)(1 - z^{-1}q^n)}. \tag{1.25}$$

By putting $z = 1$ in (1.25) we find

$$\sum_{m=-\infty}^{\infty} N_V(m, n) = p(n). \tag{1.26}$$

Our main results are

$$\begin{aligned} N_V(0, 5, 5n + 4) &= N_V(1, 5, 5n + 4) = \cdots \\ &= N_V(4, 5, 5n + 4) = p(5n + 4)/5, \end{aligned} \tag{1.27}$$

$$\begin{aligned} N_V(0, 7, 7n + 5) &= N_V(1, 7, 7n + 5) = \cdots \\ &= N_V(6, 7, 7n + 5) = p(7n + 5)/7, \end{aligned} \tag{1.28}$$

$$(1.29)\qquad N_V(0,11,11n+6) = N_V(1,11,11n+6) = \cdots$$
$$= N_V(10,11,11n+6) = p(11n+6)/11.$$

We give a direct proof of these results in §2.

Incredible as it may seem, (1.7) and (1.27) follow from two identities (see (1.30) and (1.31) below) that appear in Ramanujan's "lost" notebook. For an introduction to the "lost" notebook see Andrews [3]. We state these two identities as they appear:

$$(1.30)\qquad F(q^{1/5}) = A(q) - 4q^{1/5}\cos^2\frac{2n\pi}{5}B(q) + 2q^{2/5}\cos\frac{4n\pi}{5}C(q)$$
$$-2q^{3/5}\cos\frac{2n\pi}{5}D(q),$$

and

$$(1.31)\qquad f(q^{1/5}) = \left\{A(q) - 4\sin^2\frac{n\pi}{5}\phi(q)\right\} + q^{1/5}B(q) + 2q^{2/5}\cos\frac{2n\pi}{5}C(q)$$
$$-2q^{3/5}\cos\frac{2n\pi}{5}\left\{D(q) + 4\sin^2\frac{2n\pi}{5}\frac{\psi(q)}{q}\right\},$$

where $n = 1, 2$ and

$$(1.32)\qquad F(q) = \frac{(1-q)(1-q^2)(1-q^3)\cdots}{\left(1-2q\cos\frac{2n\pi}{5}+q^2\right)\left(1-2q^2\cos\frac{2n\pi}{5}+q^4\right)\cdots}$$

$$(1.33)\qquad f(q) = 1 + \frac{q}{\left(1-2q\cos\frac{2n\pi}{5}+q^2\right)}$$
$$+\frac{q^4}{\left(1-2q\cos\frac{2n\pi}{5}+q^2\right)\left(1-2q^2\cos\frac{2n\pi}{5}+q^4\right)} + \cdots$$

$$(1.34)\qquad A(q) = \frac{1-q^2-q^3+q^9+\cdots}{(1-q)^2(1-q^4)^2(1-q^6)^2\cdots},$$

$$(1.35)\qquad B(q) = \frac{(1-q^5)(1-q^{10})(1-q^{15})\cdots}{(1-q)(1-q^4)(1-q^6)\cdots},$$

$$(1.36)\qquad C(q) = \frac{(1-q^5)(1-q^{10})(1-q^{15})\cdots}{(1-q^2)(1-q^3)(1-q^7)\cdots},$$

$$(1.37)\qquad D(q) = \frac{1-q-q^4+q^7+\cdots}{(1-q^2)^2(1-q^3)^2(1-q^7)^2\cdots},$$

$$\phi(q) = -1 + \left\{ \frac{1}{1-q} + \frac{q^5}{(1-q)(1-q^4)(1-q^6)} + \frac{q^{20}}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})} + \cdots \right\}, \tag{1.38}$$

$$\psi(q) = -1 + \left\{ \frac{1}{1-q^2} + \frac{q^5}{(1-q^2)(1-q^3)(1-q^7)} + \frac{q^{20}}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})} + \cdots \right\}. \tag{1.39}$$

We note the appearance of the functions $A(q)$, $B(q)$, $C(q)$, $D(q)$ in both (1.30) and (1.31). There seems no simple explanation for this curious fact. In §3 we prove (1.30). In §4 we show how (1.30) leads to the following relations for the crank of vector partitions:

$$N_V(1,5,5n) = N_V(2,5,5n), \tag{1.40}$$

$$N_V(0,5,5n+1) + N_V(1,5,5n+1) = 2N_V(2,5,5n+1), \tag{1.41}$$

$$N_V(0,5,5n+2) = N_V(1,5,5n+2), \tag{1.42}$$

$$N_V(0,5,5n+3) = N_V(2,5,5n+3). \tag{1.43}$$

In §§5 and 6 we derive identities similar to (1.30) but involving 7 respectively 11 instead of 5. This enables us to obtain the following relations for the crank of vector partitions:

$$N_V(1,7,7n) = N_V(2,7,7n) = N_V(3,7,7n), \tag{1.44}$$

$$N_V(0,7,7n+1) + N_V(1,7,7n+1) = 2N_V(2,7,7n+1), \tag{1.45}$$

$$N_V(2,7,7n+1) = N_V(3,7,7n+1), \tag{1.46}$$

$$N_V(0,7,7n+2) = N_V(1,7,7n+2) = N_V(3,7,7n+2), \tag{1.47}$$

$$\begin{aligned} N_V(0,7,7n+3) &= N_V(3,7,7n+3), \\ N_V(1,7,7n+3) &= N_V(2,7,7n+3), \end{aligned} \tag{1.48}$$

$$N_V(0,7,7n+4) = N_V(2,7,7n+4) = N_V(3,7,7n+4), \tag{1.49}$$

$$\begin{aligned} N_V(0,7,7n+6) &= N_V(2,7,7n+6), \\ N_V(1,7,7n+6) &= N_V(3,7,7n+6), \end{aligned} \tag{1.50}$$

$$\begin{aligned} N_V(1,11,11n) &= N_V(2,11,11n) = N_V(3,11,11n) \\ &= N_V(4,11,11n) = N_V(5,11,11n), \end{aligned} \tag{1.51}$$

$$N_V(0,11,11n+1) + N_V(1,11,11n+1) = 2N_V(2,11,11n+1), \tag{1.52}$$

$$\begin{aligned} N_V(2,11,11n+1) &= N_V(3,11,11n+1) \\ &= N_V(4,11,11n+1) = N_V(5,11,11n+1), \end{aligned} \tag{1.53}$$

$$(1.54)\qquad N_V(0,11,11n+2) = N_V(1,11,11n+2) = N_V(3,11,11n+2) = N_V(4,11,11n+2) = N_V(5,11,11n+2),$$

$$(1.55)\qquad N_V(0,11,11n+3) = N_V(3,11,11n+3),$$

$$(1.56)\qquad N_V(1,11,11n+3) = N_V(2,11,11n+3) = N_V(4,11,11n+3) = N_V(5,11,11n+3),$$

$$(1.57)\qquad N_V(0,11,11n+4) = N_V(2,11,11n+4) = N_V(4,11,11n+4),$$

$$(1.58)\qquad N_V(1,11,11n+4) = N_V(3,11,11n+4) = N_V(5,11,11n+4),$$

$$(1.59)\qquad N_V(0,11,11n+5) = N_V(1,11,11n+5) = N_V(3,11,11n+5) = N_V(5,11,11n+5),$$

$$(1.60)\qquad N_V(2,11,11n+5) = N_V(4,11,11n+5),$$

$$(1.61)\qquad N_V(0,11,11n+7) = N_V(2,11,11n+7) = N_V(3,11,11n+7) = N_V(5,11,11n+7),$$

$$(1.62)\qquad N_V(1,11,11n+7) = N_V(4,11,11n+7),$$

$$(1.63)\qquad N_V(0,11,11n+8) = N_V(2,11,11n+8) = N_V(5,11,11n+8),$$

$$(1.64)\qquad N_V(1,11,11n+8) = N_V(3,11,11n+8) = N_V(4,11,11n+8),$$

$$(1.65)\qquad N_V(0,11,11n+9) = N_V(4,11,11n+9),$$

$$(1.66)\qquad N_V(1,11,11n+9) = N_V(2,11,11n+9) = N_V(3,11,11n+9) = N_V(5,11,11n+9),$$

$$(1.67)\qquad N_V(0,11,11n+10) = N_V(1,11,11n+10) = N_V(2,11,11n+10) = N_V(4,11,11n+10) = N_V(5,11,11n+10).$$

In §7 we derive other forms for the generating functions for $N(m,n)$ and $N_V(m,n)$. In §8 we show not only how (1.7) follows from (1.31) but also we are able to show that (1.31) is actually equivalent to the main result of Atkin and Swinnerton-Dyer's paper [7, Theorem 4]. As well as we are able to prove the following inequalities for the rank of ordinary partitions:

$$(1.68)\qquad N(1,5,5n) > N(2,5,5n) \quad \text{for } n \geqslant 1,$$

$$(1.69)\qquad N(2,5,5n+3) > N(0,5,5n+3) \quad \text{for } n \geqslant 3,$$

$$(1.70)\qquad N(0,7,7n) + N(1.7,7n) > 2N(2,7,7n) \quad \text{for } n \geqslant 0,$$

$$(1.71)\qquad N(3,7,7n+2) > N(2,7,7n+2) \quad \text{for } n \geqslant 8,$$

$$(1.72)\qquad N(0,7,7n+6) > N(3,7,7n+6) \quad \text{for } n \geqslant 5.$$

We introduce some standard notation.

$$(1.73)\qquad (a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

$$(1.74)\qquad (a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a)_n, \quad \text{where } |q| < 1.$$

In §7 we need the basic hypergeometric function

$$(1.75)\qquad {}_m\Phi_n\left[\begin{matrix} a_1, \ldots, a_m; q, z \\ b_1, \ldots, b_n \end{matrix}\right] = \sum_{j \geqslant 0} \frac{(a_1)_j (a_2)_j \cdots (a_m)_j z^j}{(b_1)_j (b_2)_j \cdots (b_n)_j (q)_j},$$

where $|z| < 1$, $|q| < 1$ and $b_i \neq q^{-n}$ for any nonnegative integer n.

2. A direct proof of the main result. We first note that (1.27)–(1.29) can be written more compactly as

(2.1)

$$N_V(0, t, tn + \delta_t) = N_V(1, t, tn + \delta_t) = \cdots = N_V(t - 1, t, tn + \delta_t) = \frac{p(tn + \delta_t)}{t}$$

for $t = 5, 7, 11$ where δ_t is the reciprocal of 24 modulo t. We need the following elementary but fundamental lemma:

LEMMA (2.2). *For t prime*, (2.1) *is equivalent to the coefficient of* $q^{tn+\delta_t}$ *in*

$$(2.3)\qquad \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - \zeta_t q^n)(1 - \zeta_t^{-1} q^n)}$$

being zero, where

$$(2.4)\qquad \zeta_t = \exp(2\pi i/t).$$

PROOF. Let t be prime. First we write (2.3) in terms of $N_V(k, t, n)$. Substituting $z = \zeta_t$ into the left-hand side of (1.25) we have

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m, n) \zeta_t^m q^n &= \sum_{k=0}^{t-1} \sum_{\substack{m \equiv k \\ (\mathrm{mod}\, t)}} \sum_{n=0}^{\infty} N_V(m, n) \zeta_t^m q^n \\ &= \sum_{k=0}^{t-1} \zeta_t^k \sum_{n=0}^{\infty} \left(\sum_{\substack{m \equiv k \\ (\mathrm{mod}\, t)}} N_V(m, n) \right) q^n \\ &= \sum_{k=0}^{t-1} \zeta_t^k \sum_{n=0}^{\infty} N_V(k, t, n) q^n, \quad \text{by (1.22)}. \end{aligned}$$

Hence we have

$$(2.5)\qquad \frac{(q)_\infty}{(\zeta_t q)_\infty (\zeta_t^{-1} q)_\infty} = \sum_{k=0}^{t-1} \zeta_t^k \sum_{n=0}^{\infty} N_V(k, t, n) q^n$$

and we find that

$$\sum_{k=0}^{t-1} N_V(k, t, tn + \delta_t) \zeta_t^k$$

is equal to the coefficient of $q^{tn+\delta_t}$ in $(q)_\infty/(\zeta_t q)_\infty (\zeta_t^{-1} q)_\infty$. Now, suppose (2.1) is true then the coefficient of $q^{tn+\delta_t}$ in $(q)_\infty/(\zeta_t q)_\infty (\zeta_t^{-1} q)_\infty$ is

$$\sum_{k=0}^{t-1} N_V(k, t, tn + \delta_t) \zeta_t^k = N_V(0, t, tn + \delta_t) \sum_{k=0}^{t-1} \zeta_t^k = 0,$$

as required. Conversely, suppose that the coefficient of $q^{tn+\delta_t}$ in $(q)_\infty/(\zeta_t q)_\infty(\zeta_t^{-1}q)_\infty$ is zero, then

$$\sum_{k=0}^{t-1} N_V(k,t,tn+\delta_t)\zeta_t^k = 0. \tag{2.6}$$

We note that the left-hand side of (2.6) is a polynomial in ζ_t over $\mathbf{Z}$. It follows that

$$N_V(0,t,tn+\delta_t) = N_V(1,t,tn+\delta_t) = \cdots = N_V(t-1,t,tn+\delta_t) \tag{2.7}$$

since t is prime and the minimal polynomial for ζ_t over $\mathbf{Q}$ is

$$p(x) = 1 + x + x^2 + \cdots + x^{t-1}. \tag{2.8}$$

Finally, from (1.26) and (2.7) we have

$$p(tn+\delta_t) = \sum_{k=0}^{t-1} N_V(k,t,tn+\delta_t) = tN_V(0,t,tn+\delta_t)$$

and (2.1) follows. □

We can now proceed with the proof of (2.1) (for $t = 5, 7, 11$). There are three cases. The cases $t = 5$ and 7 depend only on classical identities of Euler and Jacobi. The proof for the case $t = 11$ is analogous but depends on an identity due to Winquist [24].

Case 1. $t = 5$. We need

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = 1 + \sum_{n=1}^{\infty}(-1)^n q^{n(3n-1)/2}(1+q^n) \quad \text{(Euler)} \tag{2.9}$$

and

$$\prod_{n=1}^{\infty}(1-q^n)(1-zq^n)(1-z^{-1}q^n) = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} z^{-n}\left(\frac{1-z^{2n+1}}{1-z}\right). \tag{2.10}$$

The latter follows easily from Jacobi's triple product identity:

$$\prod_{n=1}^{\infty}(1-q^n)(1-zq^n)(1-z^{-1}q^{n-1}) = \sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n(n+1)/2}. \tag{2.11}$$

Now, from (2.9) and (2.10) we have

(2.12)

$$\frac{(q)_\infty}{(\zeta_5 q)_\infty(\zeta_5^{-1}q)_\infty} = \frac{(q)_\infty\{(q)_\infty(\zeta_5^2 q)_\infty(\zeta_5^{-2}q)_\infty\}}{(q)_\infty(\zeta_5 q)_\infty(\zeta_5^{-1}q)_\infty(\zeta_5^2 q)_\infty(\zeta_5^{-2}q)_\infty}$$

$$= \frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}\sum_{m=0}^{\infty}(-1)^m q^{m(m+1)/2}(\zeta_5^{-2})^m\left(1-(\zeta_5^2)^{2m+1}\right)/(1-\zeta_5^2)}{(q^5;q^5)_\infty}$$

$$\left(\text{since } 1-x^5 = \prod_{k=-2}^{2}(1-\zeta_5^k x)\right)$$

$$= \frac{\sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m} q^{n(3n-1)/2+m(m+1)/2}\zeta_5^{-2m}\left(\left(1-\zeta_5^{2(2m+1)}\right)/(1-\zeta_5^2)\right)}{(q^5;q^5)_\infty}.$$

Since $n(3n-1)/2 \equiv 0,1,2 \pmod 5$ and $m(m+1)/2 \equiv 0,1,3 \pmod 5$, the power of q is congruent to 4 modulo 5 only when $n(3n-1)/2 \equiv 1$ and $m(m+1)/2 \equiv 3 \pmod 5$ in which case $m \equiv 2 \pmod 5$ so that $2m+1 \equiv 0 \pmod 5$ and the coefficient of q^{5n+4} in (2.12) is zero, as required.

Case 2. $t = 7$. Similarly,

(2.13)

$$\frac{(q)_\infty}{(\zeta_7 q)_\infty(\zeta_7^{-1}q)_\infty} = \frac{\{(q)_\infty(\zeta_7^2 q)_\infty(\zeta_7^{-2}q)_\infty\}\{(q)_\infty(\zeta_7^3 q)_\infty(\zeta_7^{-3}q)_\infty\}}{(q)_\infty(\zeta_7 q)_\infty(\zeta_7^{-1}q)_\infty(\zeta_7^2 q)_\infty(\zeta_7^{-2}q)_\infty(\zeta_7^3 q)_\infty(\zeta_7^{-3}q)_\infty}$$

$$= \frac{\sum_{n,m\geqslant 0}(-1)^{n+m}q^{n(n+1)/2+m(m+1)/2}\zeta_7^{-2n-3m}\left(\dfrac{1-\zeta_7^{2(2n+1)}}{1-\zeta_7^2}\right)\left(\dfrac{1-\zeta_7^{3(2m+1)}}{1-\zeta_7^3}\right)}{(q^7;q^7)_\infty}.$$

Since $n(n+1)/2 \equiv 0,1,3,6 \pmod 7$, the power of q is congruent to 5 modulo 7 only when $n(n+1)/2 \equiv m(m+1)/2 \equiv 6 \pmod 7$ in which case $n \equiv m \equiv 3 \pmod 7$ and the coefficient of q^{7n+5} in (2.13) is zero, as required.

Case 3. $t = 11$. We need Winquist's identity

(2.14)

$$\prod_{n=1}^{\infty}(1-q^n)^2(1-yq^{n-1})(1-y^{-1}q^n)(1-zq^{n-1})(1-z^{-1}q^n)$$
$$\cdot(1-yz^{-1}q^{n-1})(1-y^{-1}zq^n)(1-yzq^{n-1})(1-y^{-1}z^{-1}q^n)$$
$$= \sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j}\{(y^{-3i}-y^{3i+3})(z^{-3j}-z^{3j+1})$$
$$+(y^{-3j+1}-y^{3j+2})(z^{3i+2}-z^{-3i-1})\}q^{3i(i+1)/2+j(3j+1)/2}$$

Substituting $y = \zeta_{11}^9$ and $z = \zeta_{11}^5$ in (2.14) yields

(2.15)

$$(1-\zeta_{11}^9)(1-\zeta_{11}^5)(1-\zeta_{11}^4)(1-\zeta_{11}^3)(q)_\infty^2(\zeta_{11}^2 q)_\infty(\zeta_{11}^{-2}q)_\infty(\zeta_{11}^5 q)_\infty(\zeta_{11}^{-5}q)_\infty$$
$$\cdot(\zeta_{11}^4 q)_\infty(\zeta_{11}^{-4}q)_\infty(\zeta_{11}^3 q)_\infty(\zeta_{11}^{-3}q)_\infty$$
$$= \sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j}\{(\zeta_{11}^{6i}-\zeta_{11}^{5i+5})(\zeta_{11}^{7j}-\zeta_{11}^{4j+5})$$
$$+(\zeta_{11}^{6j+9}-\zeta_{11}^{5j+7})(\zeta_{11}^{4i+10}-\zeta_{11}^{7i+6})\}q^{3i(i+1)/2+j(3j+1)/2}.$$

so that

(2.16)

$$\frac{(q)_\infty}{(\zeta_{11}q)_\infty(\zeta_{11}^{-1}q)_\infty} = \frac{(q)_\infty^2\prod_{2\leqslant k\leqslant 5}(\zeta_{11}^k q)_\infty(\zeta_{11}^{-k}q)_\infty}{(q)_\infty\prod_{1\leqslant k\leqslant 5}(\zeta_{11}^k q)_\infty(\zeta_{11}^{-k}q)_\infty}$$
$$= \frac{1}{(1-\zeta_{11}^3)(1-\zeta_{11}^4)(1-\zeta_{11}^5)(1-\zeta_{11}^9)(q^{11};q^{11})_\infty}$$
$$\cdot\sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j}\{(\zeta_{11}^{6i}-\zeta_{11}^{5i+5})(\zeta_{11}^{7j}-\zeta_{11}^{4j+5})$$
$$+(\zeta_{11}^{6j+9}-\zeta_{11}^{5j+7})(\zeta_{11}^{4i+10}-\zeta_{11}^{7i+6})\}q^{3i(i+1)/2+j(3j+1)/2}.$$

Since $3i(i+1)/2 \equiv 0, 1, 3, 7, 8, 9 \pmod{11}$ and $j(3j+1)/2 \equiv 0, 1, 2, 4, 5, 7 \pmod{11}$, the power of q is congruent to 6 modulo 11 only when $i \equiv 5$ and $j \equiv 9 \pmod{11}$ in which case $6i \equiv 5i + 5, 7j \equiv 4j + 5, 6j + 9 \equiv 5j + 7, 4i + 10 \equiv 7i + 6 \pmod{11}$ and the coefficient of q^{11n+6} in (2.16) is zero, as required. □

3. An identity from Ramanujan's "lost" notebook. In this section we derive (1.30). Throughout this section $\zeta = \zeta_5 = \exp(2\pi i/5)$. After replacing q by q^5 we see that (1.30) is equivalent to

$$F(q) = A(q^5) - (\zeta^n + \zeta^{-n})^2 qB(q^5) + (\zeta^{2n} + \zeta^{-2n})q^2C(q^5) - (\zeta^n + \zeta^{-n})q^3D(q^5), \tag{3.1}$$

where $A(q), \ldots, D(q)$ are defined in (1.34)–(1.37),

$$F(q) = F_n(q) = \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1-\zeta^n q^m)(1-\zeta^{-n}q^m)} \quad \text{(from (1.32))} \tag{3.2}$$

and $n = 1, 2$. First we show that there is no loss of generality if we assume $n = 1$. If either side of (3.1) is expanded as a power series about $q = 0$ it is clear that the coefficients are elements of $\mathbf{Q}(\zeta)$ and the case $n = 2$ is obtainable from the case $n = 1$ via the $\mathbf{Q}$-automorphism given by $\zeta \mapsto \zeta^2$. Likewise the case $n = 1$ follows from the case $n = 2$ by considering the $\mathbf{Q}$-automorphism given by $\zeta \mapsto \zeta^3$. For $n = 1$ (3.1) is

$$F(q) = A(q^5) + (\zeta + \zeta^{-1} - 1)qB(q^5) - (\zeta + \zeta^{-1} + 1)q^2C(q^5) - (\zeta + \zeta^{-1})q^3D(q^5), \tag{3.3}$$

since $-(\zeta + \zeta^{-1})^2 = -(\zeta^2 + 2 + \zeta^{-2}) = \zeta + \zeta^{-1} - 1$, $\zeta^2 + \zeta^{-2} = -(\zeta + \zeta^{-1} + 1)$. Next we note that each of $A(q), \ldots, D(q)$ can be written in terms of infinite products. Now, from (1.34)

$$\begin{aligned} A(q) &= \frac{1 - q^2 - q^3 + q^9 + \cdots}{(1-q)^2(1-q^4)^2(1-q^6)^2 \cdots} \\ &= \frac{\sum_{m=-\infty}^{\infty}(-1)^m q^{m(5m-1)/2}}{\prod_{n=1}^{\infty}(1-q^{5n-4})^2(1-q^{5n-1})^2} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^{5n-3})(1-q^{5n-2})(1-q^{5n})}{(1-q^{5n-4})^2(1-q^{5n-1})^2}, \end{aligned} \tag{3.4}$$

by Jacobi's triple product identity (2.11). Similarly we find that

$$B(q) = \prod_{n=1}^{\infty} \frac{(1-q^{5n})}{(1-q^{5n-4})(1-q^{5n-1})}, \tag{3.5}$$

$$C(q) = \prod_{n=1}^{\infty} \frac{(1-q^{5n})}{(1-q^{5n-3})(1-q^{5n-2})}, \tag{3.6}$$

$$D(q) = \prod_{n=1}^{\infty} \frac{(1-q^{5n-4})(1-q^{5n-1})(1-q^{5n})}{(1-q^{5n-3})^2(1-q^{5n-2})^2}. \tag{3.7}$$

Hence we see that (1.30) is equivalent to

$$(3.8)\quad \prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-\zeta q^n)(1-\zeta^{-1}q^n)}$$
$$=\prod_{n=1}^{\infty}\frac{(1-q^{25n-15})(1-q^{25n-10})(1-q^{25n})}{(1-q^{25n-20})^2(1-q^{25n-5})^2}$$
$$+(\zeta+\zeta^{-1}-1)q\prod_{n=1}^{\infty}\frac{(1-q^{25n})}{(1-q^{25n-20})(1-q^{25n-5})}$$
$$-(\zeta+\zeta^{-1}+1)q^2\prod_{n=1}^{\infty}\frac{(1-q^{25n})}{(1-q^{25n-15})(1-q^{25n-10})}$$
$$-(\zeta+\zeta^{-1})q^3\prod_{n=1}^{\infty}\frac{(1-q^{25n-20})(1-q^{25n-5})(1-q^{25n})}{(1-q^{25n-15})^2(1-q^{25n-10})^2}.$$

OBSERVATION. Since no term involving q^{5n+4} appears on the right-hand side of (3.8) we note that (1.27) is a corollary of (3.8) by Lemma (2.2). It was this observation that led us originally to consider (1.27). (3.8) follows easily from two lemmas.

LEMMA (3.9).

(3.10)
$$\prod_{n=1}^{\infty}\frac{1}{(1-\zeta q^n)(1-\zeta^{-1}q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{25n-20})(1-q^{25n-5})}$$
$$+(\zeta+\zeta^{-1})q\prod_{n=1}^{\infty}\frac{1}{(1-q^{25n-15})(1-q^{25n-10})}$$

where $\zeta=\exp(2\pi i/5)$.

PROOF. We need Jacobi's triple product in a different form:

$$(3.11)\quad \sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n^2}=(zq;q^2)_\infty(z^{-1}q;q^2)_\infty(q^2;q^2)_\infty.$$

(3.11) can be obtained from (2.11) by simply replacing q and q^2 and z by zq^{-1}.

$$(3.12)\quad \frac{1}{(\zeta q)_\infty(\zeta^{-1}q)_\infty}=\frac{(q)_\infty(\zeta^2q)_\infty(\zeta^{-2}q)_\infty}{(q)_\infty(\zeta q)_\infty(\zeta^{-1}q)_\infty(\zeta^2q)_\infty(\zeta^{-2}q)_\infty}$$
$$=\frac{1}{(1-\zeta^{-2})(q^5;q^5)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\zeta^{2n}q^{(n^2+n)/2},$$

by (2.11). Since $(n^2+n)/2\equiv 0,1,3 \pmod 5$ we can write

$$(3.13)\quad \sum_{n=-\infty}^{\infty}(-1)^n\zeta^{2n}q^{(n^2+n)/2}=F_0(q)+F_1(q)+F_3(q),$$

where $F_i(q)$ contains those terms on the left-hand side of (3.13) in which the power of q is congruent to i modulo 5.

$$(3.14)\quad F_0(q) = \sum_{\substack{(r^2+r)/2\equiv 0 \\ (\mathrm{mod}\ 5)}} (-1)^r \zeta^{2r} q^{(r^2+r)/2}$$

$$= \sum_{\substack{r\equiv 0,4 \\ (\mathrm{mod}\ 5)}} (-1)^r \zeta^{2r} q^{(r^2+r)/2}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^{5n} \zeta^{10n} q^{(25n^2+5n)/2} + \sum_{n=-\infty}^{\infty} (-1)^{5n-1} \zeta^{10n-2} q^{(25n^2-5n)/2}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{(25n^2-5n)/2} - \zeta^{-2} \sum_{n=-\infty}^{\infty} (-1)^n q^{(25n^2-5n)/2}$$

(replacing n by $-n$ in the first sum)

$$= (1-\zeta^{-2})(q^{10};q^{25})_\infty (q^{15};q^{25})_\infty (q^{25};q^{25})_\infty,$$

by replacing q by $q^{25/2}$ and z by $q^{-5/2}$ in (3.11). Similarly we find that

$$(3.15)\quad F_1(q) = (\zeta-\zeta^2)q \sum_{n=-\infty}^{\infty} (-1)^n q^{(25n^2-15n)/2}$$

$$= (\zeta-\zeta^2)q(q^5;q^{25})_\infty (q^{20};q^{25})_\infty (q^{25};q^{25})_\infty$$

and

$$(3.16)\quad F_3(q) = \zeta^4 q^3 \sum_{n=-\infty}^{\infty} (-1)^n q^{(25n^2+25n)/2}$$

$$= 0.$$

By writing $\sum_{n=-\infty}^{\infty}(-1)^n\zeta^{2n}q^{(n^2+n)/2}$ in terms of our expressions for the $F_i(q)$ and substituting this into (3.12) we find that

$$\prod_{n=1}^{\infty} \frac{1}{(1-\zeta q^n)(1-\zeta^{-1}q^n)}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{25n})}{(1-q^{5n})} \left\{ \prod_{n=1}^{\infty} (1-q^{25n-15})(1-q^{25n-10}) + q\frac{(\zeta-\zeta^2)}{1-\zeta^{-2}} \prod_{n=1}^{\infty} (1-q^{25n-20})(1-q^{25n-5}) \right\}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{25n-20})(1-q^{25n-5})} + (\zeta+\zeta^{-1})q \prod_{n=1}^{\infty} \frac{1}{(1-q^{25n-15})(1-q^{25n-10})}$$

which is (3.10) since $(1-\zeta^{-2})(\zeta+\zeta^{-1}) = \zeta-\zeta^2$. Here we have used

$$(3.17)\quad (q^5;q^5)_\infty = (q^5;q^{25})_\infty (q^{10};q^{25})_\infty (q^{15};q^{25})_\infty (q^{20};q^{25})_\infty (q^{25}q^{25})_\infty. \quad \square$$

The following lemma is well known. It is (2.1) in Watson [23] and it is also a special case of Lemma 6 in Atkin and Swinnerton-Dyer [7]. It is interesting to note that (3.19) also follows from an identity of M. Hirschhorn. In fact if we replace q by $q^{5/2}$, a by $-q^{3/2}$ and b by $-q^{1/2}$ in (2.1) of Hirschhorn [17] we obtain (3.19) after dividing both sides of the resulting identity by (3.17).

LEMMA (3.18).

$$\prod_{n=1}^{\infty}(1-q^n)=\prod_{n=1}^{\infty}(1-q^{25n})\left(\prod_{n=1}^{\infty}\frac{(1-q^{25n-15})(1-q^{25n-10})}{(1-q^{25n-20})(1-q^{25n-5})}-q-q^2\prod_{n=1}^{\infty}\frac{(1-q^{25n-20})(1-q^{25n-5})}{(1-q^{25n-15})(1-q^{25n-10})}\right). \tag{3.19}$$

We can now proceed with the proof of (3.8). From (3.10) and (3.19) we have

$$\begin{aligned}
\frac{(q)_\infty}{(\zeta q)_\infty(\zeta^{-1}q)_\infty}
&=(q^{25};q^{25})_\infty\left(\frac{(q^{15};q^{25})_\infty(q^{10};q^{25})_\infty}{(q^{20};q^{25})_\infty(q^{5};q^{25})_\infty}-q-q^2\frac{(q^{20};q^{25})_\infty(q^{5};q^{25})_\infty}{(q^{10};q^{25})_\infty(q^{15};q^{25})_\infty}\right)\\
&\quad\times\left(\frac{1}{(q^{5};q^{25})_\infty(q^{20};q^{25})_\infty}+(\zeta+\zeta^{-1})q\frac{1}{(q^{10};q^{25})_\infty(q^{15};q^{25})_\infty}\right)\\
&=\frac{(q^{10};q^{25})_\infty(q^{15};q^{25})_\infty(q^{25};q^{25})_\infty}{(q^{5};q^{25})_\infty^2(q^{20};q^{25})_\infty^2}+(\zeta+\zeta^{-1}-1)q\frac{(q^{25};q^{25})_\infty}{(q^{5};q^{25})_\infty(q^{20};q^{25})_\infty}\\
&\quad-(\zeta+\zeta^{-1}+1)q^2\frac{(q^{25};q^{25})_\infty}{(q^{10};q^{25})_\infty(q^{15};q^{25})_\infty}\\
&\quad-(\zeta+\zeta^{-1})q^3\frac{(q^{5};q^{25})_\infty(q^{20};q^{25})_\infty(q^{25};q^{25})_\infty}{(q^{10};q^{25})_\infty^2(q^{15};q^{25})_\infty^2}
\end{aligned} \tag{3.20}$$

which is (3.8). □

4. Some results for vector partitions modulo 5. We define the following generating functions:

$$_VR_b(d)={}_VR_b(d,t)=\sum_{n\geqslant 0}N_V(b,t,tn+d)q^n \tag{4.1}$$

and

$$_VR_{b,c}(d)={}_VR_{b,c}(d,t)={}_VR_b(d)-{}_VR_c(d). \tag{4.2}$$

These functions are analogous to Atkin and Swinnerton-Dyer's $R_b(d)$ and $R_{b,c}(d)$ (see (8.1) and (8.2) below). For convenience we write

$$A(q)=\sum_{n\geqslant 0}a_nq^n=\prod_{n=1}^{\infty}\frac{(1-q^{5n-3})(1-q^{5n-2})(1-q^{5n})}{(1-q^{5n-4})^2(1-q^{5n-1})^2}, \tag{4.3}$$

$$B(q) = \sum_{n\geqslant 0} b_n q^n = \prod_{n=1}^{\infty} \frac{(1-q^{5n})}{(1-q^{5n-4})(1-q^{5n-1})}, \tag{4.4}$$

$$C(q) = \sum_{n\geqslant 0} c_n q^n = \prod_{n=1}^{\infty} \frac{(1-q^{5n})}{(1-q^{5n-3})(1-q^{5n-2})}, \tag{4.5}$$

$$D(q) = \sum_{n\geqslant 0} d_n q^n = \prod_{n=1}^{\infty} \frac{(1-q^{5n-4})(1-q^{5n-1})(1-q^{5n})}{(1-q^{5n-3})^2(1-q^{5n-2})^2}. \tag{4.6}$$

THEOREM (4.7). *For* $t = 5$,

$$_VR_{0,1}(0) = A(q), \tag{4.8}$$

$$_VR_{0,1}(1) = -2B(q), \tag{4.9}$$

$$_VR_{1,2}(1) = B(q), \tag{4.10}$$

$$_VR_{1,2}(2) = -C(q), \tag{4.11}$$

$$_VR_{0,1}(3) = -{}_VR_{1,2}(3) = D(q), \tag{4.12}$$

and all other functions $_VR_{b,b+1}(d)$, *where* $b = 0$ *or* 1, *are zero.*

REMARK. (1.40)–(1.43) follow from Theorem (4.7).

PROOF. From (2.5) and (3.2) we can write (3.3) as

(4.13)

$$\sum_{k=0}^{4} \zeta^k \sum_{n=0}^{\infty} N_V(k,5,n) q^n = A(q^5) + (\zeta + \zeta^{-1} - 1) q B(q^5) - (\zeta + \zeta^{-1} + 1) q^2 C(q^5) - (\zeta + \zeta^{-1}) q^3 D(q^5),$$

where $\zeta = \exp(2\pi i/5)$. Picking out those terms in which the power of q is congruent to 0 modulo 5 we obtain

$$\sum_{k=0}^{4} \zeta^k \sum_{n=0}^{\infty} N_V(k,5,5n) q^{5n} = \sum_{n=0}^{\infty} a_n q^{5n}. \tag{4.14}$$

Picking out the coefficient of q^{5n} we have

$$\sum_{k=0}^{4} \zeta^k N_V(k,5,5n) = a_n \tag{4.15}$$

or

$$(N_V(0,5,5n) - a_n) + N_V(1,5,5n)\zeta + N_V(2,5,5n)\zeta^2 + N_V(3,5,5n)\zeta^3 + N_V(4,5,5n)\zeta^4 = 0. \tag{4.16}$$

Since a_n and the $N_V(k,5,5n)$ are rational integers it follows that

(4.17)

$$N_V(0,5,5n) - a_n = N_V(1,5,5n) = N_V(2,5,5n) = N_V(3,5,5n) = N_V(4,5,5n).$$

Hence we have

$$\sum_{n=0}^{\infty} (N_V(0,5,5n) - N_V(1,5,5n)) q^n = \sum_{n=0}^{\infty} a_n q^n = A(q), \tag{4.18}$$

which is (4.8) and

$$(4.19)\qquad {}_VR_{1,2}(0) = \sum_{n=0}^{\infty} (N_V(1,5,5n) - N_V(2,5,5n))q^n = 0.$$

Similarly the remaining results also follow from (4.13). □

5. Some results for vector partitions modulo 7. The main result of this section is Theorem (5.1). It is an identity similar to (1.30) but involving 7 instead of 5. Its proof is completely analogous to that of Lemma (3.9). This result does not appear in Ramanujan's "lost" notebook.

THEOREM (5.1).

(5.2)

$$\prod_{n=1}^{\infty} \frac{(1-q^n)}{\left(1 - 2\cos\frac{2\pi}{7}q^n + q^{2n}\right)}$$
$$= \prod_{n=1}^{\infty}(1-q^{7n})\Big\{X^2(q^7) + \left(2\cos\frac{2\pi}{7} - 1\right)qX(q^7)Y(q^7) + 2\cos\frac{4\pi}{7}q^2Y^2(q^7)$$
$$+\left(2\cos\frac{6\pi}{7}+1\right)q^3X(q^7)Z(q^7)$$
$$-2\cos\frac{2\pi}{7}q^4Y(q^7)Z(q^7) - \left(2\cos\frac{4\pi}{7}+1\right)q^6Z^2(q^7)\Big\},$$

where

$$(5.3)\qquad X(q) = \prod_{\substack{n=1\\ n\not\equiv 0,\pm 3\\ (\mathrm{mod}\ 7)}}^{\infty} (1-q^n)^{-1},\quad Y(q) = \prod_{\substack{n=1\\ n\not\equiv 0,\pm 2\\ (\mathrm{mod}\ 7)}}^{\infty} (1-q^n)^{-1},$$

$$Z(q) = \prod_{\substack{n=1\\ n\not\equiv 0,\pm 1\\ (\mathrm{mod}\ 7)}}^{\infty} (1-q^n)^{-1}.$$

OBSERVATION. Since no term involving q^{7n+5} appears on the right-hand side of (5.2) we note that (1.28) is a corollary of Theorem (5.1) by Lemma (2.2). By using arguments analogous to that of §4 we find that Theorem (5.1) allows calculation of the ${}_VR_{b,c}(d,7)$ (defined in (4.2)).

THEOREM (5.4). *For* $t = 7$,

$$(5.5)\qquad {}_VR_{0,1}(0) = X^2(q)\prod_{n=1}^{\infty}(1-q^n),$$

$$(5.6)\qquad {}_VR_{0,1}(1) = -2X(q)Y(q)\prod_{n=1}^{\infty}(1-q^n),$$

$$(5.7)\qquad {}_VR_{1,2}(1) = X(q)Y(q)\prod_{n=1}^{\infty}(1-q^n),$$

$$(5.8)\qquad {}_VR_{1,2}(2) = -{}_VR_{2,3}(2) = -Y^2(q)\prod_{n=1}^{\infty}(1-q^n),$$

$$(5.9) \qquad {}_VR_{0,1}(3) = -{}_VR_{2,3}(3) = X(q)Z(q)\prod_{n=1}^{\infty}(1-q^n),$$

$$(5.10) \qquad {}_VR_{0,1}(4) = -{}_VR_{1,2}(4) = Y(q)Z(q)\prod_{n=1}^{\infty}(1-q^n),$$

$$(5.11) \qquad {}_VR_{0,1}(6) = -{}_VR_{1,2}(6) = {}_VR_{2,3}(6) = -Z^2(q)\prod_{n=1}^{\infty}(1-q^n),$$

where $X(q)$, $Y(q)$ *and* $Z(q)$ *are defined in* (5.3), *and all other functions* ${}_VR_{b,b+1}(d)$, *where* $0 \leqslant b \leqslant 2$, *are zero.*

REMARK. (1.44)–(1.50) follow directly from Theorem (5.4).

6. Some results for vector partitions modulo 11. The main result of this section is Theorem (6.7). It is an identity similar to (1.30) but involving 11 instead of 5. We will provide a sketch of the proof. To describe our results neatly we introduce some more notation. For $t > 1$, $1 \leqslant a < t$, let

$$(6.1) \qquad P(a,t) = \prod_{n=1}^{\infty}(1-q^{11(tn+a-t)})(1+q^{11(tn-a)})(1-q^{11tn})$$

and for $1 \leqslant b < 33$, let

$$(6.2) \qquad Q(b) = P(b,33).$$

We need some preliminary results. The proofs of Lemmas (6.3) and (6.5) are analogous to that of Lemma (3.9). In this section ζ refers to any primitive 11th root of unity.

LEMMA (6.3). *If* $\zeta^{11} = 1$, $\zeta \neq 1$ *then*

$$(6.4) \qquad \prod_{n=1}^{\infty}(1-q^n)(1-\zeta q^{n-1})(1-\zeta^{-1}q^n)$$
$$= (1-\zeta)P(5,11) + (\zeta^2-\zeta^{-1})qP(4,11) + (\zeta^{-2}-\zeta^3)q^3P(3,11)$$
$$+ (\zeta^4-\zeta^{-3})q^6P(2,11) + (\zeta^{-4}-\zeta^5)q^{10}P(1,11).$$

where $P(a,t)$ *is defined in* (6.1).

LEMMA (6.5). *If* $\zeta^{11} = 1$, $\zeta \neq 1$ *then*

$$(6.6) \qquad \prod_{n=1}^{\infty}(1-\zeta q^{3n-2})(1-\zeta^{-1}q^{3n-1})(1-q^{3n})$$
$$= \big(Q(16) + \zeta^4q^{22}Q(5)\big) - \zeta q\big(Q(14) + \zeta^2q^{11}Q(8)\big)$$
$$-\zeta^{-1}q^2\big(Q(13) + \zeta^6q^{33}Q(2)\big) - \zeta^{-3}q^{15}\big(Q(7) - \zeta^{-1}q^{11}Q(4)\big)$$
$$+\zeta^2q^5\prod_{n=1}^{\infty}(1-q^{121n}) + \zeta^{-2}q^7\big(Q(10) - \zeta^{-3}q^{33}Q(1)\big).$$

where $Q(b)$ *is defined in* (6.2).

THEOREM (6.7).

(6.8)

$$\prod_{n=1}^{\infty} \frac{(1-q^n)}{\left(1-2\cos\frac{2\pi}{11}q^n+q^{2n}\right)}$$

$$= \prod_{n=1}^{\infty}(1-q^{11n})^{-1}\Big\{(A_0B_0-q^{33}A_7B_4)+\left(2\cos\frac{2\pi}{11}-1\right)q(A_0B_1-q^{44}A_8B_4)$$
$$+2\cos\frac{4\pi}{11}q^2(A_0B_2-q^{22}A_9B_4)$$
$$+\left(2\cos\frac{6\pi}{11}+1\right)q^3(A_3B_0-q^{22}A_7B_7)$$
$$+\left(2\cos\frac{4\pi}{11}+2\cos\frac{8\pi}{11}+1\right)q^4(A_3B_1-q^{33}A_8B_7)$$
$$-\left(2\cos\frac{4\pi}{11}+2\cos\frac{8\pi}{11}\right)q^5(A_3B_2-q^{11}A_8B_7)$$
$$+\left(2\cos\frac{2\pi}{11}+2\cos\frac{8\pi}{11}\right)q^7(A_0B_7-q^{11}A_3B_4)$$
$$-\left(2\cos\frac{4\pi}{11}+2\cos\frac{10\pi}{11}+1\right)q^{19}(A_7B_1-q^{11}A_8B_0)$$
$$-\left(2\cos\frac{8\pi}{11}+1\right)q^9(A_9B_0-q^{11}A_7B_2)$$
$$-2\cos\frac{6\pi}{11}q^{10}(A_9B_1-q^{22}A_8B_2)\Big\},$$

where

(6.9) $A_0=Q(15),\quad A_3=Q(12),\quad A_7=Q(6),\quad A_8=Q(3),\quad A_9=Q(9),$

(6.10) $$B_0=Q(16)-q^{22}Q(5),$$

(6.11) $$B_1=Q(14)-q^{11}Q(8),$$

(6.12) $$B_2=Q(13)-q^{33}Q(2),$$

(6.13) $$B_4=Q(7)+q^{11}Q(4),$$

(6.14) $$B_7=Q(10)+q^{33}Q(1).$$

OBSERVATION. Since no term involving q^{11n+6} appears on the right-hand side of (6.8) we note that (1.29) is a corollary of Theorem (6.7) by Lemma (2.2).

PROOF. We first observe as in Hirschhorn **[18]** that Winquist's identity (2.14) can be written as

(6.15)

$$(a)_\infty(a^{-1}q)_\infty(b)_\infty(b^{-1}q)_\infty(ab)_\infty(a^{-1}b^{-1}q)_\infty(ab^{-1})_\infty(a^{-1}bq)_\infty(q)_\infty^2$$
$$=(a^3;q^3)_\infty(a^{-3}q^3;q^3)_\infty(q^3;q^3)_\infty$$
$$\cdot\{(b^3q;q^3)_\infty(b^{-3}q^2;q^3)_\infty(q^3;q^3)_\infty-b(b^3q^2;q^3)_\infty(b^{-3}q;q^3)_\infty(q^3;q^3)_\infty\}$$
$$-ab^{-1}(b^3;q^3)_\infty(b^{-3}q^3;q^3)_\infty(q^3;q^3)_\infty$$
$$\cdot\{(a^3q;q^3)_\infty(a^{-3}q^2;q^3)_\infty(q^3;q^3)_\infty-a(a^3q^2;q^3)_\infty(a^{-3}q;q^3)_\infty(q^3;q^3)_\infty\}.$$

We proceed as in Case 3, §2. After substituting $a = \zeta^9$ and $b = \zeta^5$ in (6.15) we find that

(6.16)

$$\frac{(q)_\infty}{(\zeta q)_\infty(\zeta^{-1}q)_\infty} = (1-\zeta^3)^{-1}(1-\zeta^4)^{-1}(1-\zeta^5)^{-1}(1-\zeta^9)^{-1}(q^{11};q^{11})_\infty^{-1}$$
$$\times\{(\zeta^5;q^3)_\infty(\zeta^{-5}q^3;q^3)_\infty(q^3;q^3)_\infty((\zeta^4q;q^3)_\infty(\zeta^{-4}q^2;q^3)_\infty(q^3;q^3)_\infty$$
$$-\zeta^5(\zeta^4q^2;q^3)_\infty(\zeta^{-4}q;q^3)_\infty(q^3;q^3)_\infty)$$
$$-\zeta^4(\zeta^4;q^3)_\infty(\zeta^{-4}q^3;q^3)_\infty(q^3;q^3)_\infty((\zeta^5q;q^3)_\infty(\zeta^{-5}q^2;q^3)_\infty(q^3;q^3)_\infty$$
$$-\zeta^9(\zeta^5q^2;q^3)_\infty(\zeta^{-5}q;q^3)_\infty(q^3;q^3)_\infty)\}$$

By using (6.4) and (6.6) we find, after some simplification, that the right-hand side of (6.16) reduces to the right-hand side of (6.8). □

By using arguments analogous to that of §4 we find that Theorem (6.7) allows calculation of the ${}_VR_{b,c}(d,11)$ (defined in (4.2)).

THEOREM (6.17). *For* $t = 11$,

(6.18) $${}_VR_{0,1}(0) = \prod_{n=1}^{\infty}(1-q^n)^{-1}(A_0^*B_0^* - q^3A_7^*B_4^*),$$

(6.19) $${}_VR_{0,1}(1) = -2\prod_{n=1}^{\infty}(1-q^n)^{-1}(A_0^*B_1^* - q^4A_8^*B_4^*),$$

(6.20) $${}_VR_{1,2}(1) = \prod_{n=1}^{\infty}(1-q^n)^{-1}(A_0^*B_1^* - q^4A_8^*B_4^*),$$

(6.21) $${}_VR_{1,2}(2) = -{}_VR_{2,3}(2) = -\prod_{n=1}^{\infty}(1-q^n)^{-1}(A_0^*B_2^* - q^2A_9^*B_4^*),$$

(6.22) $${}_VR_{0,1}(3) = -{}_VR_{2,3}(3) = {}_VR_{3,4}(3) = \prod_{n=1}^{\infty}(1-q^n)^{-1}(A_3^*B_0^* - q^2A_7^*B_7^*),$$

(6.23) $${}_VR_{0,1}(4) = -{}_VR_{1,2}(4) = {}_VR_{2,3}(4) = -{}_VR_{3,4}(4) = {}_VR_{4,5}(4)$$
$$= \prod_{n=1}^{\infty}(1-q^n)^{-1}(A_3^*B_1^* - q^3A_8^*B_7^*),$$

(6.24) $${}_VR_{1,2}(5) = -{}_VR_{2,3}(5) = {}_VR_{3,4}(5) = -{}_VR_{4,5}(5)$$
$$= \prod_{n=1}^{\infty}(1-q^n)^{-1}(A_3^*B_2^* - qA_9^*B_7^*),$$

(6.25) $${}_VR_{0,1}(7) = -{}_VR_{1,2}(7) = {}_VR_{3,4}(7) = -{}_VR_{4,5}(7)$$
$$= -\prod_{n=1}^{\infty}(1-q^n)^{-1}(A_0^*B_7^* - qA_3^*B_4^*),$$

(6.26) $${}_VR_{0,1}(8) = -{}_VR_{1,2}(8) = {}_VR_{2,3}(8) = -{}_VR_{4,5}(8)$$
$$= -q\prod_{n=1}^{\infty}(1-q^n)^{-1}(A_7^*B_1^* - qA_8^*B_0^*),$$

(6.27) $${}_VR_{0,1}(9) = -{}_VR_{3,4}(9) = {}_VR_{4,5}(9) = -\prod_{n=1}^{\infty}(1-q^n)^{-1}\left(A_9^*B_0^* - qA_7^*B_2^*\right),$$

(6.28) $${}_VR_{2,3}(10) = -{}_VR_{3,4}(10) = \prod_{n=1}^{\infty}(1-q^n)^{-1}\left(A_9^*B_1^* - q^2A_8^*B_2^*\right),$$

where $A_i^* = A_i(q^{1/11})$, $B_j^*(q) = B_j(q^{1/11})$ *and the* A_i *and* B_j *are defined in* (6.9)–(6.14), *and all other functions* ${}_VR_{b,b+1}(d)$, *where* $0 \leqslant b \leqslant 4$, *are zero.*

REMARK. (1.51)–(1.67) follow directly from Theorem (6.17).

7. Generating function identifies for the rank and the crank. In this section we show how Ramanujan's identity, (1.31), is related to the rank of ordinary partitions. We find generating function identities for $N(m,n)$ and show how these are related to the results of Dyson, Atkin and Swinnerton-Dyer. We set up the results needed in the next section and finally we find that the generating function of $N_V(m,n)$ has a form similar to that of $N(m,n)$.

Euler has proved

(7.1) $$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2} = \prod_{n=1}^{\infty}\frac{1}{1-q^n} = \sum_{n=0}^{\infty}p(n)q^n.$$

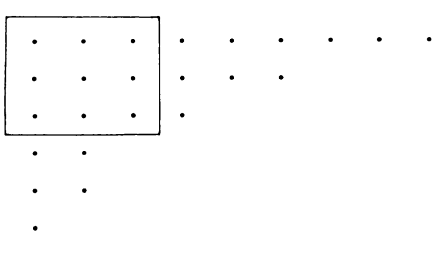
By utilizing the concept of the Durfee square, Sylvester [**21**] has demonstrated (7.1) combinatorially, For a discussion of his proof and its generalizations see Andrews [**4**]. We show Sylvester's argument is easily modified to obtain

(7.2)

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(1-zq)(1-zq^2)\cdots(1-zq^n)(1-z^{-1}q)(1-z^{-1}q^2)\cdots(1-z^{-1}q^n)} = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}N(m,n)z^mq^n.$$

Note. Here we are taking $N(0,0) = 1$. This differs from Atkin and Swinnerton-Dyer who take $N(0,0) = 0$. Also (7.1) is (7.2) with $z = 1$.

For each partition π we find the largest square (starting from the upper left-hand corner) of dots contained in its graphical representation. This square is called the *Durfee square* (after W. P. Durfee). For example, if π is the partition $9 + 6 + 4 + 2 + 2 + 1 + 1$, then its graphical representation is



and the 3×3 "Durfee" square is indicated. Thus each partition π with Durfee square of side s can be written $\pi = s^2 + \pi_1 + \pi_2$ where π_1 is the partition (whose parts are all $\leqslant s$) of nodes below the Durfee square and π_2 is the partition (whose parts are all $\leqslant s$) gotten by reading off the columns of nodes to the right of the

Durfee square. The rank of π is the length of the first row minus the length of the first column or the number of columns to the right of the Durfee square minus the number of rows below the Durfee square. That is,

$$\operatorname{rank}(\pi) = \#(\pi_1) - \#(\pi_2).$$

Hence, if we let $N_s(m, n)$ denote the number of partitions of n with Durfee square of side s and rank m we obtain

$$(7.3) \qquad q^{s^2} \cdot \frac{1}{(1 - zq) \cdots (1 - zq^s)} \cdot \frac{1}{(1 - z^{-1}q) \cdots (1 - z^{-1}q^s)} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_s(m, n) z^m q^n.$$

Thus if we sum over all s we obtain (7.2). We now see that Ramanujan's identity, (1.31), is telling us something about the rank since the left-hand side of (1.31) is the left-hand side of (7.2) with $z = \exp(2\pi i/5)$. This is explored further in the next section.

The following identity found by Dyson **[13]** is proved in Atkin and Swinnerton-Dyer's paper. For $m > 0$,

$$(7.4) \qquad \sum_{n=0}^{\infty} N(m, n) q^n = \prod_{k=1}^{\infty} (1 - q^k)^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2 + mn}(1 - q^n).$$

It is interesting to note that (7.4) follows from (7.2) and Watson's **[22]** q-analog of Whipple's theorem:

$$(7.5) \qquad {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-N}; q, \dfrac{a^2q^{N+2}}{bcde} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{N+1} \end{matrix}\right] = \frac{(aq)_N (aq/de)_N}{(aq/d)_N (aq/e)_N} {}_4\Phi_3\left[\begin{matrix} aq/bc, d, e, q^{-N}; q, q \\ deq^{-N}/a, aq/b, aq/c \end{matrix}\right].$$

In fact, if $|q| < 1$, $|q| < |z| < |q|^{-1}$, and we let $b = z$, $c = z^{-1}$, $a = 1$ and let $d, e, N \to \infty$ we obtain

(7.6)

$$\begin{aligned}
(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n (z^{-1}q)_n}
&= 1 + \sum_{n=1}^{\infty} \frac{(1 - z)(1 - z^{-1})(-1)^n q^{n(3n+1)/2}(1 + q^n)}{(1 - zq^n)(1 - z^{-1}q^n)} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n q^{n(3n-1)/2}(1 + q^n) \frac{q^n(1 - z)(1 - z^{-1})}{(1 - zq^n)(1 - z^{-1}q^n)} \\
&= 1 + \sum_{n=1}^{\infty} (-1)^n q^{n(3n-1)/2}(1 + q^n) \left\{1 - \left(\frac{1 - q^n}{1 + q^n}\right)\left(\frac{1}{1 - zq^n} + \frac{1}{1 - z^{-1}q^n} - 1\right)\right\} \\
&= 1 + \sum_{n=1}^{\infty} \Bigg[(-1)^n q^{n(3n-1)/2}(1 + q^n) \\
&\qquad \times \left\{1 - \frac{1 - q^n}{1 + q^n} \sum_{m=0}^{\infty} z^m q^{nm} - \frac{1 - q^n}{1 + q^n} \sum_{m=1}^{\infty} z^{-m} q^{mn}\right\}\Bigg].
\end{aligned}$$

Hence, if we assume $m > 0$ then by (7.2) and (7.6) we have

$$(7.7)\quad \sum_{n=0}^{\infty} N(m,n)q^n = \text{coefficient of } z^m \text{ in } \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}$$

$$= \prod_{k=1}^{\infty}(1-q^k)^{-1}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n-1)/2+mn}(1-q^n),$$

which is the desired result. Similarly we find that

$$(7.8)\quad \sum_{n=0}^{\infty} N(0,n)q^n = 1 + \prod_{k=1}^{\infty}(1-q^k)^{-1}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n-1)/2}(1-q^n).$$

The following lemma is needed in the next section. It does not appear in Atkin and Swinnerton-Dyer's paper.

LEMMA (7.9). *For* $|q| < 1$, $|q| < |z| < |q|^{-1}$, $z \neq 1$,

$$(7.10)\quad -1 + \frac{1}{1-z}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n} = \frac{z}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{3n(n+1)/2}}{1-zq^n}.$$

PROOF. From (7.6) we have

(7.11)

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}$$

$$= (q)_\infty^{-1}\left\{1 + \sum_{n=1}^{\infty}(-1)^n q^{n(3n-1)/2}\frac{(1+q^n)q^n(1-z)(1-z^{-1})}{(1-zq^n)(1-z^{-1}q^n)}\right\}$$

$$= (q)_\infty^{-1}\left\{1 + \sum_{n=1}^{\infty}(-1)^n q^{n(3n-1)/2}(1+q^n)\right.$$

$$\left.\times\left[1 - \frac{1-q^n}{(1+q^n)(1-zq^n)} - \frac{(1-q^n)z^{-1}q^n}{(1+q^n)(1-z^{-1}q^n)}\right]\right\}$$

$$= 1 + (q)_\infty^{-1}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n-1)/2}(1-q^n)\left\{\frac{1}{1-zq^n} + \frac{z^{-1}q^n}{1-z^{-1}q^n}\right\}$$

(by (2.9))

$$= 1 + \frac{z^{-1}}{(q)_\infty}\sum{}'(-1)^{n-1}q^{n(3n+1)/2}\frac{1-q^n}{1-z^{-1}q^n}.$$

Here Σ' means $\Sigma_{n=-\infty;n\neq 0}^{\infty}$. Replacing z by z^{-1} in (7.11) we obtain
(7.12)

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n} &= 1+z(q)_\infty^{-1}\sum{}'(-1)^{n-1}q^{n(3n+1)/2}\frac{1-q^n}{1-zq^n}\\ &= 1+z(q)_\infty^{-1}\left\{\sum{}'(-1)^{n-1}\frac{q^{n(3n+1)/2}}{1-zq^n}+\sum{}'(-1)^n\frac{q^{3n(n+1)/2}}{1-zq^n}\right\}\\ &= 1+z(q)_\infty^{-1}\left\{\sum{}'(-1)^{n-1}\frac{q^{n(3n+1)/2}}{1-zq^n}+\sum{}'(-1)^n q^{n(3n+1)/2}\right.\\ &\qquad\left.+1-(q)_\infty+\sum{}'(-1)^n\frac{q^{3n(n+1)/2}}{1-zq^n}\right\}\qquad\text{(by (2.9))}\\ &= 1-z+z(q)_\infty^{-1}\left\{1+\sum{}'(-1)^n\frac{q^{3n(n+1)/2}}{1-zq^n}\right.\\ &\qquad\left.-\sum{}'(-1)^n\frac{q^{n(3n+1)/2}}{1-zq^n}(1-(1-zq^n))\right\}\\ &= (1-z)+z(q)_\infty^{-1}\left\{1+(1-z)\sum{}'(-1)^n\frac{q^{3n(n+1)/2}}{1-zq^n}\right\}.\end{aligned}$$

Hence,

(7.13)
$$\begin{aligned}&-1+\frac{1}{1-z}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq)_n(z^{-1}q)_n}\\ &= -1+\frac{1}{1-z}\left[(1-z)+\frac{z}{(q)_\infty}\left\{1+(1-z)\sum{}'(-1)^n\frac{q^{3n(n+1)/2}}{1-zq^n}\right\}\right]\\ &= \frac{1}{(q)_\infty}\left\{\frac{z}{1-z}+z\sum{}'(-1)^n\frac{q^{3n(n+1)/2}}{1-zq^n}\right\}\\ &= \frac{z}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{3n(n+1)/2}}{1-zq^n},\end{aligned}$$

which is (7.10). □

Dyson [**13**] also conjectured that the generating function for the crank should have a form analogous to (7.4), and indeed it does (see Theorem (7.19) below). We need the limiting form of Jackson's theorem (Andrews [**1**, Theorem 3.2]):
(7.14)

$$\begin{aligned}&{}_6\phi_5\left[\begin{matrix}z, q\sqrt{z}, -q\sqrt{z}, a_1, a_2, a_3; q, zq/a_1a_2a_3\\ \sqrt{z}, -\sqrt{z}, \frac{zq}{a_1}, \frac{zq}{a_2}, \frac{zq}{a_3}\end{matrix}\right]\\ &= \prod_{n=1}^{\infty}\frac{(1-zq^n)(1-za_1^{-1}a_2^{-1}q^n)(1-za_1^{-1}a_3^{-1}q^n)(1-za_2^{-1}a_3^{-1}q^n)}{(1-za_1^{-1}q^n)(1-za_2^{-1}q^n)(1-za_3^{-1}q^n)(1-za_1^{-1}a_2^{-1}a_3^{-1}q^n)}\end{aligned}$$

If we let $z \to 1$, $a_3 \to \infty$, $a_1 = z$, $a_2 = z^{-1}$ then (7.14) becomes

$$(7.15) \qquad 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)} = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-zq^n)(1-z^{-1}q^n)} \quad (\text{cf. } (7.6)).$$

We note as in Andrews [5] that (7.15) can also be obtained by calculating the classical partial fractions decomposition of the right-hand side. We also note that in the same paper Andrews has used a different form of (7.15) to obtain expansions of certain infinite products in terms of Hecke modular forms.

We proceed as in (7.6) to obtain

$$(7.16) \qquad \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-zq^n)(1-z^{-1}q^n)} = 1 + \sum_{n=1}^{\infty} (-1)^n q^{n(n-1)/2}(1+q^n) \cdot \left\{1 - \frac{1-q^n}{1+q^n} \sum_{m=0}^{\infty} z^m q^{nm} - \frac{1-q^n}{1+q^n} \sum_{m=1}^{\infty} z^{-m} q^{mn}\right\}.$$

Hence, if $m \geqslant 1$ we find

$$(7.17) \qquad \sum_{n=0}^{\infty} N_V(m,n) q^n = \text{coefficient of } z^m \text{ in } \prod_{n=1}^{\infty} \frac{1-q^n}{(1-zq^n)(1-z^{-1}q^n)} = \prod_{k=1}^{\infty} (1-q^k)^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2+nm}(1-q^n).$$

Similarly, we find that

$$(7.18) \qquad \sum_{n=0}^{\infty} N_V(0,n) q^n = \prod_{k=1}^{\infty} (1-q^k)^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2}(1-q^n).$$

We note the last expression in (7.17) is the same as that in (7.4) except that "3" has been replaced by "1". In a later paper we shall investigate what happens when "3" is replaced by an arbitrary odd integer. After combining (7.17), (7.18) and using (1.23) we have the following theorem.

THEOREM (7.19).

$$(7.20) \qquad \sum_{n=0}^{\infty} N_V(m,n) q^n = \prod_{k=1}^{\infty} (1-q^k)^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2+n|m|}(1-q^n).$$

REMARK. We note that (7.20) is a useful form for calculating the coefficients $N_V(m,n)$. In §10 we give a table for small values of n.

An examination of this table leads us to conjecture the result:

$$N_V(m,n) \geqslant 0 \quad \text{for } (m,n) \neq (0,1). \tag{7.21}$$

This is a surprising result since one would expect the sign of $N_V(m,n)$ to be random. A proof will appear in a later paper.

8. A second identity from the "lost" notebook and some inequalities for the rank modulo 5 and 7. In this section we show Ramanujan's identity, (1.31), and the main result of Atkin and Swinnerton-Dyer's paper [7, Theorem 4] (see Theorem (8.3) below) are equivalent. As well we derive inequalities, (1.68)–(1.72), for the rank of ordinary partitions modulo 5 and 7.

Following Atkin and Swinnerton-Dyer we define

$$R_b(d) = R_b(d,t) = \sum_{n=0}^{\infty} N(b,t,tn+d)q^n \tag{8.1}$$

and

$$R_{b,c}(d) = R_{b,c}(d,t) = R_b(d) - R_c(d). \tag{8.2}$$

THEOREM (8.3) (ATKIN AND SWINNERTON-DYER). *For* $t = 5$,

$$R_{1,2}(0) = q\prod_{n=1}^{\infty}(1-q^{5n})^{-1}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{15n(n+1)/2}}{1-q^{5n+1}}, \tag{8.4}$$

$$R_{0,2}(0) + 2R_{1,2}(0) = A(q), \tag{8.5}$$

$$R_{0,2}(1) = B(q), \tag{8.6}$$

$$R_{1,2}(1) = 0, \tag{8.7}$$

$$R_{0,2}(2) = 0, \tag{8.8}$$

$$R_{1,2}(2) = C(q), \tag{8.9}$$

$$R_{0,2}(3) = -q\prod_{n=1}^{\infty}(1-q^{5n})^{-1}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{15n(n+1)/2}}{1-q^{5n+2}}, \tag{8.10}$$

$$R_{0,1}(3) + R_{0,2}(3) = D(q), \tag{8.11}$$

$$R_{0,2}(4) = 0, \tag{8.12}$$

$$R_{1,2}(4) = 0, \tag{8.13}$$

where $A(q)$, $B(q)$, $C(q)$ *and* $D(q)$ *are defined in* (4.3)–(4.6).

Recall that $\phi(q)$ and $\psi(q)$ are defined in (1.38) and (1.39), respectively. For convenience we write

$$\phi(q) = \sum_{n=0}^{\infty}\phi_n q^n = -1 + \frac{1}{1-q}\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q^4;q^5)_n(q^6;q^5)_n}, \tag{8.14}$$

$$\frac{\psi(q)}{q} = \sum_{n=0}^{\infty}\psi_n q^n = \frac{1}{q}\left\{-1 + \frac{1}{1-q^2}\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q^3;q^5)_n(q^7;q^5)_n}\right\}. \tag{8.15}$$

THEOREM (8.16). (1.31) and Theorem (8.3) are equivalent.

PROOF. Let $\zeta = \exp(2\pi i/5)$. After replacing q by q^5 (1.31) becomes

(8.17)
$$\sum_{m=0}^{\infty} \frac{q^{m^2}}{(\zeta^n q)_m(\zeta^{-n} q)_m} = A(q^5) + (\zeta^n + \zeta^{-n} - 2)\phi(q^5) + qB(q^5) + (\zeta^n + \zeta^{-n})q^2 C(q^5) - (\zeta^n + \zeta^{-n})q^3\left\{D(q^5) - (\zeta^{2n} + \zeta^{-2n} - 2)\frac{\psi(q^5)}{q^5}\right\},$$

for $n = 1, 2$. Without loss of generality we may suppose that $n = 1$ by using an argument analogous to that used at the beginning of §3.

We now write the left-hand side of (8.17) in terms of $N(m, n)$. From (7.2) we have

(8.18)
$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta q)_n(\zeta^{-1} q)_n} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n)\zeta^m q^n = \sum_{n=0}^{\infty} \sum_{k=0}^{4} \zeta^k \sum_{\substack{m\equiv k \\ (\mathrm{mod}\, 5)}} N(m, n) q^n = \sum_{n=0}^{\infty} \sum_{k=0}^{4} \zeta^k N(k, 5, n) q^n.$$

Hence (1.31) is equivalent to

(8.19)
$$\sum_{n=0}^{\infty} \sum_{k=0}^{4} \zeta^k N(k, 5, n) q^n = A(q^5) + (\zeta + \zeta^{-1} - 2)\phi(q^5) + qB(q^5) + (\zeta + \zeta^{-1})q^2 C(q^5) - (\zeta + \zeta^{-1})q^3\left\{D(q^5) - (\zeta^2 + \zeta^{-2} - 2)\frac{\psi(q^5)}{q^5}\right\}.$$

Now suppose (1.31) is true. Picking out the coefficient of q^{5n} on both sides of (8.19) we obtain

(8.20)
$$\sum_{k=0}^{4} \zeta^k N(k, 5, 5n) = a_n + (\zeta + \zeta^{-1} - 2)\phi_n \quad \text{(by (4.3) and (8.14))},$$

or

(8.21)
$$\begin{aligned}(N(0, 5, 5n) - a_n + 2\phi_n) + (N(1, 5, 5n) - \phi_n)\zeta + N(2, 5, 5n)\zeta^2 \\ + N(3, 5, 5n)\zeta^3 + (N(4, 5, 5n) - \phi_n)\zeta^4 = 0.\end{aligned}$$

Since the coefficients of the ζ^k are rational integers it follows that

(8.22)
$$N(0, 5, 5n) - a_n + 2\phi_n = N(1, 5, 5n) - \phi_n = N(2, 5, 5n).$$

Hence,

$$(8.23)\quad \begin{aligned} R_{1,2}(0) &= \sum_{n=0}^{\infty} (N(1,5,5n) - N(2,5,5n))q^n \\ &= \sum_{n=0}^{\infty} \phi_n q^n = \phi(q) \\ &= -1 + \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^4;q^5)_n (q^6;q^5)_n} \\ &= q \prod_{n=1}^{\infty} (1-q^{5n})^{-1} \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{15n(n+1)/2}}{1-q^{5n+1}} \end{aligned}$$

(by Lemma (7.9) with q replaced by q^5 and z by q),

which is (8.4), and

$$(8.24)\quad \begin{aligned} &R_{0,2}(0) + 2R_{1,2}(0) \\ &= \sum_{n=0}^{\infty} (N(0,5,5n) - N(2,5,5n) + 2(N(1,5,5n) - N(2,5,5n)))q^n \\ &= \sum_{n=0}^{\infty} (a_n - 2\phi_n + 2\phi_n)q^n \\ &= A(q), \end{aligned}$$

which is (8.5). Similarly (8.6)–(8.13) follow from (8.19) noting that

$$(8.25)\quad \begin{aligned} \frac{\psi(q)}{q} &= \frac{1}{q}\left\{-1 + \frac{1}{1-q^2} \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^3;q^5)_n (q^7;q^5)_n}\right\} \\ &= q \prod_{n=1}^{\infty} (1-q^{5n})^{-1} \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{15n(n+1)/2}}{1-q^{5n+2}}, \end{aligned}$$

by Lemma (7.9) with q replaced by q^5 and z by q^2. We see that Theorem (8.3) follows from (1.31). By reversing the arguments above, (1.31) follows from Theorem (8.3) and the two are equivalent. □

We now turn to the inequalities (1.68)–(1.72). From (8.23) we have

$$(8.26)\quad R_{1,2}(0,5) = \phi(q) = \frac{q}{1-q} + \sum_{n=1}^{\infty} \frac{q^{5n^2}}{(1-q)(q^4;q^5)_n (q^6;q^5)_n}$$

so that

$$(8.27)\quad R_{1,2}(0,5) - \frac{q}{1-q} = \sum_{n=1}^{\infty} \frac{q^{5n^2}}{(1-q)(q^4;q^5)_n (q^6;q^5)_n}$$

and since the power series expansion of the right-hand side of (8.27) clearly has nonnegative coefficients we have (1.68). Similarly (1.69) follows by considering

(8.28)

$$R_{2,0}(3,5) - \frac{q^3}{1-q} = -\frac{q^3}{1-q} + \frac{q}{1-q^2} + \sum_{n=1}^{\infty} \frac{q^{5n^2-1}}{(1-q^2)(q^3;q^5)_n(q^7;q^5)_n}$$

(by (8.25) and (8.10))

$$= q + \frac{q^4}{1-q^2}\left\{\frac{1}{(1-q^3)(1-q^7)} - 1\right\} + \sum_{n=2}^{\infty} \frac{q^{5n^2-1}}{(1-q^2)(q^3;q^5)_n(q^7;q^5)_n}.$$

(1.70)–(1.72) follow analogously from Atkin and Swinnerton-Dyer [**7**, Theorem 5] and Lemma (7.9) by considering the first and last expression in each of the following equations:

$$(8.29)\quad R_{0,2}(0,7) + R_{1,2}(0,7) = 1 + q\prod_{n=1}^{\infty}(1-q^{7n})^{-1} \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{21n(n+1)/2}}{1-q^{7n+1}}$$

(by [**7**, (6.23) and (6.24)])

$$= \frac{1}{1-q}\sum_{n=0}^{\infty} \frac{q^{7n^2}}{(q^8;q^7)_n(q^6;q^7)_n}$$

(by Lemma (7.9) with q replaced by q^7 and z by q)

$$(8.30)\quad R_{3,2}(2,7) - \frac{q^8}{1-q}$$

$$= -\frac{q^8}{1-q} + q^2\prod_{n=1}^{\infty}(1-q^{7n})^{-1} \sum_{n=-\infty}^{\infty} (-1)^n \frac{q^{21n(n+1)/2}}{1-q^{7n+3}} \quad \text{(by [7, (6.30)])}$$

$$= -\frac{q^8}{1-q} + \frac{1}{q}\left\{-1 + \frac{1}{1-q^3}\sum_{n=0}^{\infty} \frac{q^{7n^2}}{(q^4;q^7)_n(q^{10};q^7)_n}\right\}$$

(by Lemma (7.9) with q replaced by q^7 and z by q^3)

$$= q^2 + q^5 + q^6 + \frac{q^6}{1-q^3}\left\{\frac{1}{(1-q^4)(1-q^{10})} - 1 - q^4\right\} + \sum_{n=2}^{\infty} \frac{q^{7n^2-1}}{(1-q^3)(q^4;q^7)_n(q^{10};q^7)_n},$$

(8.31) $$R_{0,3}(6,7) - \frac{q^5}{1-q}$$

$$= -\frac{q^5}{1-q} + q\prod_{n=1}^{\infty}(1-q^{7n})^{-1}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{21n(n+1)/2}}{1-q^{7n+2}} \quad \text{(by [7, (6.41)])}$$

$$= -\frac{q^5}{1-q} + \frac{1}{q}\left\{-1 + \frac{1}{1-q^2}\sum_{n=0}^{\infty}\frac{q^{7n^2}}{(q^9;q^7)_n(q^5;q^7)_n}\right\}$$

(by Lemma (7.9) with q replaced by q^7 and z by q^2)

$$= q + q^3 + \frac{q^6}{1-q^2}\left\{\frac{1}{(1-q^5)(1-q^9)} - 1\right\}$$

$$+ \sum_{n=2}^{\infty}\frac{q^{7n^2-1}}{(1-q^2)(q^9;q^7)_n(q^5;q^7)_n}.$$

Much more than (1.68)–(1.72) seems to be true. We are able to prove the following inequalities:

(8.32) $$N(0,5,5n+1) > N(1,5,5n+1) \quad \text{for } n \geqslant 0,$$

(8.33) $$N(1,5,5n+2) \geqslant N(2,5,5n+2) \quad \text{for } n \geqslant 0,$$

(8.34) $$N(0,7,7n+1) > N(1,7,7n+1) \quad \text{for } n \geqslant 0,$$

(8.35) $$N(0,7,7n+3) \geqslant N(1,7,7n+3) \quad \text{for } n \geqslant 0,$$

(8.36) $$N(1,7,7n+4) \geqslant N(2,7,7n+4) \quad \text{for } n \geqslant 0,$$

(8.37) $$N_V(1,5,5n+1) > N_V(2,5,5n+1) > N_V(0,5,5n+1) \quad \text{for } n \geqslant 0,$$

(8.38) $$N_V(2,5,5n+2) \geqslant N_V(1,5,5n+2) \quad \text{for } n \geqslant 0,$$

(8.39) $$N_V(1,7,7n+1) > N_V(2,7,7n+1) > N_V(0,7,7n+1) \quad \text{for } n \geqslant 0,$$

(8.40) $$N_V(0,7,7n+3) \geqslant N_V(1,7,7n+3) \quad \text{for } n \geqslant 0,$$

(8.41) $$N_V(0,7,7n+4) \geqslant N_V(1,7,7n+4) \quad \text{for } n \geqslant 0.$$

We leave the details until a later paper. We also conjecture the following inequalities:

CONJECTURE.

(8.42) $$N(2,5,5n) \geqslant N(0,5,5n) \quad \text{for } n \geqslant 0,$$

(8.43) $$N(0,5,5n+3) \geqslant N(1,5,5n+3) \quad \text{for } n \geqslant 0,$$

(8.44) $$N(0,7,7n) \geqslant N(1,7,7n) \geqslant N(2,7,7n) \quad \text{for } n \geqslant 7,$$

(8.45) $$N(1,7,7n+2) > N(0,7,7n+2) \quad \text{for } n \geqslant 0,$$

(8.46) $$N(2,7,7n+6) \geqslant N(1,7,7n+6) \geqslant N(0,7,7n+6) \quad \text{for } n \geqslant 2,$$

(8.47) $$N_V(0,5,5n) > N_V(1,5,5n) \quad \text{for } n \geqslant 0,$$

(8.48) $$N_V(0,5,5n+3) \geqslant N_V(1,5,5n+3) \quad \text{for } n \geqslant 2,$$

(8.49) $$N_V(0,7,7n) > N_V(1,7,7n) \quad \text{for } n \geqslant 0,$$

(8.50) $$N_V(2,7,7n+2) \geqslant N_V(0,7,7n+2) \quad \text{for } n \geqslant 0,$$

(8.51) $$N_V(1,7,7n+6) \geqslant N_V(0,7,7n+6) \quad \text{for } n \geqslant 2.$$

Note. We can prove special cases for some of the inequalities above. For instance, we can prove (8.47) for $n \equiv 1, 4 \pmod 5$.

9. Conclusion. This paper first arose from studying the two identities, (1.30) and (1.31), from Ramanujan's "lost" notebook. I would like to thank Professor G. E. Andrews for pointing out these two identities and for giving me much help and encouragement.

What makes our method work so well is that the generating function for $N_V(m,n)$ is a simple infinite product (namely (1.25)) and that plugging in $z = \zeta_p$, a primitive pth root of unity, results in an analytic function for which the dissection of the power series according to the residue of the exponent modulo p is relatively easy. Our methods can be extended to a number of problems where an infinite product like (1.25) is involved. In particular we have found the correct ranks for two-rowed plane partitions, which was asked for by Atkin **[10]** and studied by Gordon and Cheema **[12]**, and for two-colored generalized Frobenius partitions studied by Andrews **[6]** (see Garvan **[15]** for details).

10. Table. For reference we include a table for $N_V(m,n)$:

$N_V(m,n)$		m										
		0	1	2	3	4	5	6	7	8	9	10
n	0	1	0	0	0	0	0	0	0	0	0	0
	1	−1	1	0	0	0	0	0	0	0	0	0
	2	0	0	1	0	0	0	0	0	0	0	0
	3	1	0	0	1	0	0	0	0	0	0	0
	4	1	0	1	0	1	0	0	0	0	0	0
	5	1	1	0	1	0	1	0	0	0	0	0
	6	1	1	1	1	1	0	1	0	0	0	0
	7	1	2	1	1	1	1	0	1	0	0	0
	8	2	2	2	1	2	1	1	0	1	0	0
	9	2	3	2	3	1	2	1	1	0	1	0
	10	4	3	4	2	3	2	2	1	1	0	1
	11	4	5	4	4	3	3	2	2	1	1	0
	12	7	5	6	5	5	3	4	2	2	1	1
	13	7	8	7	7	5	6	3	4	2	2	1
	14	11	9	10	8	8	6	6	4	4	2	2
	15	12	13	11	12	9	9	7	6	4	4	2
	16	17	15	16	13	14	10	10	7	7	4	4
	17	19	21	18	19	15	15	11	11	7	7	4
	18	27	24	25	22	21	17	17	12	11	8	7
	19	30	33	29	29	25	24	18	18	13	11	8
	20	41	38	39	34	34	28	26	20	19	13	12

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DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

Current address: Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, Minnesota 55455